

Z-transform in Telecommunications Systems: review and applications

*Transformada Z en sistemas de Telecomunicaciones:
revisión y aplicaciones*

*Transformada Z em Sistemas de Telecomunicações:
Revisão e Aplicações*

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Abstract

Introduction: This paper explores the various applications of the **Z-Transform** in telecommunications, specifically in modulation, demodulation, and **Dual-tone Multi-frequency (DTMF)** signaling, with a focus on the **Goertzel Algorithm**.

Problem: There is a gap in understanding the specific applications of the **Z-Transform** in telecommunications engineering, especially in practical systems like DTMF detection.

Objective: The objective of this study is to analyze recent research on the application of the **Z-Transform** in telecommunications, with a particular focus on implementing **Goertzel's Algorithm** in DTMF systems.

Methodology: A bibliographic study was conducted to investigate the applications of the **Z-Transform** across various engineering fields, including control systems, automation, biomedical signal processing, and telecommunications. Additionally, a **MATLAB simulation** of the **Goertzel Algorithm** was performed and compared with the traditional **Fast Fourier Transform (FFT)** to evaluate its performance in detecting DTMF tones.

Results: The study reveals several applications of the **Z-Transform**, and the simulation demonstrates that the **Goertzel Algorithm** is more efficient than FFT in terms of processing and detecting DTMF tones.

Conclusion: The **Goertzel Algorithm** is identified as an efficient implementation of the **Z-Transform** for DTMF detection in telecommunications. It provides a viable alternative to the **FFT** in terms of computational efficiency, especially when dealing with a limited number of frequencies.

Originality: This study combines a comprehensive literature review on the **Z-Transform** and **Goertzel Algorithm**, with a comparative analysis of their efficiency in **DTMF** detection, offering new insights into their practical applications.

Limitations: The simulation was limited to a small set of frequencies, and further research is needed to assess the algorithm's performance in more complex and dynamic scenarios.

Keywords: Z-Transform, Goertzel Algorithm, Telecommunications, DTMF, Frequencies, Filters.

Resumen

Introducción: El artículo explora las diversas aplicaciones de la Transformada Z en las Telecomunicaciones, enfocándose en la modulación, demodulación y DTMF con el Algoritmo de Goertzel.

Problema: Existe una necesidad de profundizar en las aplicaciones específicas de la Transformada Z en el campo de la ingeniería particularmente en las telecomunicaciones.

Objetivo: Analizar las aplicaciones de la Transformada Z en ingeniería que se encuentran en investigaciones recientes la implementación del Algoritmo de Goertzel en sistemas DTMF.

Metodología: Se llevó a cabo un estudio bibliográfico acerca de las aplicaciones de la Transformada Z en el ámbito de la ingeniería. Se categorizaron las aplicaciones en control y automatización, procesamiento biomédico y en telecomunicaciones con DTMF se realiza una simulación del Algoritmo de Goertzel en MATLAB y se compara con FFT.

Resultados: Se encontraron variadas aplicaciones de la Transformada Z y con la simulación se comprobó la eficiencia del Algoritmo de Goertzel en términos de eficiencia para la detección de tonos DTMF.

Conclusión: El Algoritmo de Goertzel se presenta como una buena aplicación de la Transformada Z en las telecomunicaciones y es una alternativa eficiente a la FFT para la detección de tonos DTMF.

Originalidad: El estudio combina la revisión bibliográfica sobre la Transformada Z y el Algoritmo de Goertzel con un análisis comparativo de su eficiencia en DTMF.

Limitaciones: La simulación se limita a un ejemplo con pocas frecuencias para el Algoritmo de Goertzel. Se requieren estudios distintos para evaluar el rendimiento en situaciones más complejas.

Palabras clave: Transformada Z, Algoritmo de Goertzel, Telecomunicaciones, DTMF, frecuencias, filtros.

Resumo

Introdução: Este artigo explora as diversas aplicações da Transformada Z em telecomunicações, com foco em modulação, demodulação e DTMF com o algoritmo de Goertzel.

Problema: Há necessidade de se aprofundar nas aplicações específicas da Transformada Z na área de engenharia, particularmente em telecomunicações.

Objetivo: Analisar as aplicações da Transformada Z em engenharia encontradas em pesquisas recentes sobre a implementação do algoritmo de Goertzel em sistemas DTMF.

Metodologia: Foi realizada uma revisão bibliográfica sobre as aplicações da Transformada Z em engenharia. As aplicações foram categorizadas em controle e automação, processamento biomédico e telecomunicações com DTMF. Uma simulação do algoritmo de Goertzel foi realizada em MATLAB e comparada com FFT.

Resultados: Diversas aplicações da Transformada Z foram identificadas, e a simulação verificou a eficiência do algoritmo de Goertzel em termos de detecção de tons DTMF. Conclusão: O algoritmo de Goertzel apresenta-se como uma boa aplicação da transformada Z em telecomunicações e é uma alternativa eficiente à FFT para detecção de tons DTMF.

Originalidade: O estudo combina uma revisão da literatura sobre a transformada Z e o algoritmo de Goertzel com uma análise comparativa de sua eficiência em DTMF.

Limitações: A simulação limita-se a um único exemplo com poucas frequências para o algoritmo de Goertzel. Estudos separados são necessários para avaliar o desempenho em situações mais complexas.

Palavras-chave: Transformada Z, algoritmo de Goertzel, Telecomunicações, DTMF, frequências, filtros.

1. INTRODUCTION

The **Z-Transform** is a fundamental mathematical tool widely used in the analysis and design of discrete systems within the frequency domain. It originated from the **Laplace Transform**, which was developed by Pierre-Simon Laplace in the 18th century to solve differential equations in continuous systems [1]. While the **Laplace Transform** applies to continuous signals and preserves its relationship with the **Fourier Transform**, the Z-Transform is the discrete counterpart, used to analyze signals that are defined at discrete points in time.

Since the 1950s, the **Z-Transform** has become an indispensable tool in the study of discrete signals. It has numerous applications across various fields such as engineering, signal processing, and communications, providing a powerful method for analyzing and manipulating signals in the frequency domain [2].

Mathematically, the Z-Transform of a sequence $x[n]$ is defined by the Laurent series (1):

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (1)$$

where z is a complex variable and therefore $X(z)$ represents in equation (1) a function in the complex frequency domain [1].

This formulation enables the analysis of discrete signals and systems in a way that is more efficient than traditional time-domain techniques. Specifically, the **Z-Transform** simplifies the analysis of the **stability** and **frequency response of linear time-invariant (LTI) systems** [3]. By converting signals from the time domain to the frequency domain, the Z-Transform allows engineers to more easily study system behavior, stability criteria, and the impact of various system components on overall performance.

"In order for the Z-Transform to be used, it must converge, since not all possible sets or values of Z allow its convergence. The sequence of z -values at which the Z-Transform converges is called the Region of Convergence (ROC).

The Z-Transform is fundamental in the design, configuration, and analysis of digital filters in signal processing. IIR (Infinite Impulse Response) and FIR (Finite Impulse Response) filters are designed and analyzed using the Z-Transform to ensure their stability and performance in denoising or extracting specific signals [4], [5].

In an FIR filter, the Z-Transform allows for the representation of the frequency response of the filter in the z -domain and is defined by equation (2):"

$$H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} \quad (2)$$

where $h[n]$ are the filter coefficients; while an IIR filter has a representation in a rational function in Z , equation (3) [5]:

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 + \sum_{k=1}^M a_k z^{-k}} \quad (3)$$

where a_k and b_k are the filter coefficients [6].

In the field of telecommunications and control, these filters are used for tasks ranging from noise suppression to the design of robust control systems [7].

A prominent application of the Z-Transform is the Goertzel Algorithm, which is highly efficient for analyzing specific frequency components within a signal. This algorithm serves as an effective alternative to the Fast Fourier Transform (FFT) in scenarios where identifying specific frequencies is required [8]. The Goertzel Algorithm relies on a recurrence equation, enabling the calculation of the Z-Transform at specific points in the complex plane (4) [9]:

$$y[n] = x[n] + 2 * \cos(\omega_k) y[n - 1] - y[n - 2] \quad (4)$$

Where $y[n]$ represents the value of the transformed signal at time instant n , ω_k is the frequency of interest, and N is the total number of samples [2]. This algorithm is widely used in applications such as Dual-Tone Multi-Frequency (DTMF) tone dialing in telecommunication systems, where detecting specific frequencies is essential for interpreting tone signals [10].

The Z-Transform facilitates the analysis of the stability of discrete systems. A system is stable if, on the unit circle in the Z-plane, all the poles of its transfer function are present $H(z)$ [11]. This condition ensures that the response of the system does not diverge with time (Figure 1) [12], [13]. Stability is a crucial feature in the design of systems for digital control, such as PID (Proportional-Integral-Derivative) controllers, which can be analyzed and designed using the Z-Transform [14].

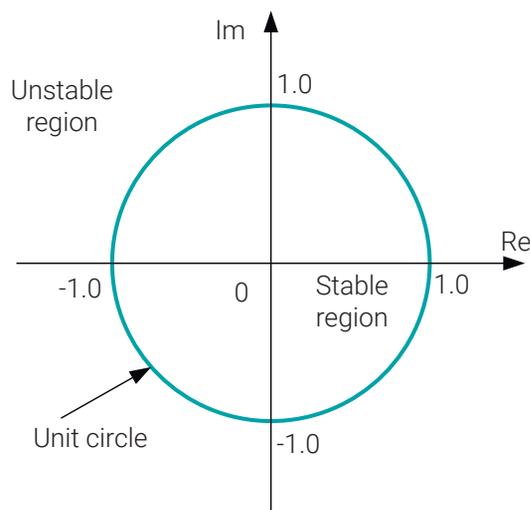


Figure 1. Z-term stability diagram [13].

In addition, the Z-Transform is also useful in the analysis of non-stationary signals. One example is the Z-Chirp Transform, which is employed in applications where signal frequencies vary rapidly over time [15], [16]. This technique provides improved resolution and accuracy in detecting frequency variations, facilitating the analysis of complex and dynamic signals [17], [18]. The Z-Chirp Transform is particularly important for analyzing systems in which frequencies change over time, offering a valuable tool for studying non-stationary signals [16].

This article presents a detailed analysis of the categorization and subcategorization of the Z-Transform, along with its specific applications in the field of telecommunications, including the use of the Goertzel Algorithm. Additionally, through a case study applied to telecommunications, a practical engineering application of this mathematical tool is demonstrated. Finally, the article concludes with key findings and insights.

2. Notation

2.1. Z-Transform

The variable z can be expressed in polar form as $z = re^{j\omega}$ where r is the modulus and ω is the phase. This allows us to study the behavior of systems and signals in the z plane [11]. The Z-transform converts a discrete sequence into a function of a complex variable, facilitating the analysis of systems (LTI) in the frequency domain [19], [3].

While the Discrete Fourier Transform-DFT (Discrete Fourier Transform) applies to finite and periodic sequences, the Z-Transform applies to infinite and non-periodic sequences. In addition, the DFT is an evaluation of the Z-Transform on the unit circle ($z = re^{j\omega}$) [4]. This can be expressed mathematically in Equation (5) from Equation (1):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]z^{-j\omega} \quad (5)$$

The relationship between the Z-Transform and the DFT is crucial in signal and system analysis because it allows the use of Fourier analysis techniques in the study of discrete systems [15].

For the Z-Transform of a sequence $x[n]$ to exist, the infinite sum in Equation (1) must converge [20]. For a sequence $x[n]$ the ROC is a ring in the Z-plane defined by two radii r_1 and r_2 [21], such that:

$$r_1 < |z| < r_2$$

The ROC is crucial for determining the causality and stability of systems [10]. A system is stable if the ROC includes the unit circle ($|z| = 1$) and it is the set of values of z for which $X(z)$ in equation (6) is finite [20].

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n)z^{-n} \right| < \infty \quad (6)$$

To return to the initial sequence $X[n]$ from its transform, the Inverse Z-Transform (7) is used.

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} dz \quad (7)$$

2.1.1. Properties of the Z-Transform

The Z-Transform has several properties that are fundamental for the analysis and design of discrete systems [12], the main properties of the Z-Transform are the following:

a. Linearity

The linearity property of the Z-Transform states that if $x_1[n]$ and $x_2[n]$ are two sequences, a and b , that are constants (8), then:

$$Z\{ax_1[n] + bx_2[n]\} = aX_1(z) + bX_2(z) \quad (8)$$

where $X_1(z)$ and $X_2(z)$ are the Z-Transforms of $x_1[n]$ and $x_2[n]$ [8].

b. Time Shifting

If a sequence $x[n]$ is shifted in time by n_0 units [12], the Z-Transform of the shifted sequence is (9):

$$Z\{x[n - n_0]\} = z^{-n_0}X(z) \quad (9)$$

c. Convolution

The Z-Transform of the convolution of two sequences $x_1[n]$ and $x_2[n]$ is the product of their Z-Transforms (10):

$$Z\{x_1[n] * x_2[n]\} = X_1(z) * bX_2(z) \quad (10)$$

d. Initial value theorem

It states that if $x[n]$ is a sequence with Z-Transform $x(z)$ then the initial value of the sequence $x[0]$ is (11):

$$x[0] = \lim_{z \rightarrow \infty} X(z) \quad (11)$$

e. The end value theorem

The final value of a sequence $x[n]$ can be determined directly from its Z-Transform if the sequence converges to a finite value (12):

$$x[\infty] = \lim_{z \rightarrow 1} (z - 1)X(z) \quad (12)$$

f. Shifting in the Z-domain

If $x[n]$ has the Z-Transform $x(z)$, then the sequence $z^k x[n]$ has the Z-Transform (13):

$$Z\{z^k x[n]\} = -z \frac{dX(z)}{dz} \quad (13)$$

g. Multiplication by an Exponential Sequence

If $x[n]$ has the Z-Transform $x(z)$, then the sequence $a^n x[n]$ has the Z-Transform (14):

$$Z\{a^n x[n]\} = X\left(\frac{z}{a}\right) \quad (14)$$

2.2. Goertzel algorithm

The Goertzel Algorithm is a technique used to evaluate the Discrete Fourier Transform (DFT) at specific frequencies, primarily in applications requiring tone detection or the identification of particular frequencies, such as in telecommunication systems and audio signal processing [22].

Developed in 1958 [23], the algorithm is named after its creator, Gerald Goertzel (1920–2002), an American computer scientist, nuclear engineer, and physicist. He was a member of the Manhattan Project at the Nuclear Development Corporation of America (NDA), an entrepreneur in the field of scientific instruments for medical use, and an IBM researcher for nearly thirty years. His groundbreaking contribution to complex system design was the use of modular structures—dividing systems into smaller, independent subsystems with distinct inputs and outputs that could be efficiently assembled and interconnected.

While the Fast Fourier Transform (FFT) computes the DFT for all frequencies, the Goertzel Algorithm is optimized for evaluating specific DFT frequencies more efficiently [24].

The Goertzel Algorithm is based on the DFT and the Z-Transform. The DFT of a discrete sequence $x[n]$ of length N is expressed by Equation (15):

$$X(k) = \sum_{n=0}^{N-1} x[n] * e^{-j\frac{2\pi}{N}kn} \quad (15)$$

Where $X(k)$ is the k -th coefficient of the DFT, $x[n]$ is the input sequence and N is the length of the sequence [25].

For this algorithm, the Z-Transform of Equation (1) is used to reformulate the DFT calculation. If we define $W_N = e^{-j\frac{2\pi}{N}}$ then $e^{-j\frac{2\pi}{N}kn} = (W_N^k)^n$. It can be expressed $X(k)$ in terms of the Z-Transform evaluated in $z = W_N^k$ (16).

$$X(k) = X(z)|_{z=W_N^k} \quad (16)$$

Where W_N^k is the slew factor for the frequency k [26].

The Algorithm furthermore reduces the number of real-valued multiplications by almost a factor of two relative to the direct DFT implementation, which is one of the advantages of its implementation and can also be easily implemented with a digital filter [27], [28].

For the filter, we use Equation (17), which can be viewed as a convolution of the input sequence $x(n)$ with the sequence $W_N^{-kn} * u(n)$ and represents the output of the filter at time n :

$$h[n] = e^{j\left(\frac{2\pi}{N}\right)kn} * u[n] = W_N^{-kn} * u[n] \quad (17)$$

Where $u(n)$ is the unit step function [25].

To analyze the filter, we have Equation (18) which represents the Z-Transform of the Goertzel filter for different frequencies.

$$h[n] = e^{j\left(\frac{2\pi}{N}\right)kn} * u[n] = W_N^{-kn} * u[n] \quad (18)$$

If the algorithm is implemented recursively, Equation (19) is obtained, which describes the update of the filter for each new sample in the input sequence considering the possible values of n . Figure 2 illustrates the flowchart of the signal for a system with impulse response $W_N^{-kn} * u[n]$ [29].

$$y_k[n] = y_k[n - 1]W_N^{-k} + x[n], \quad n = 0, 1, \dots, N, \quad (19)$$

By having the entrance $x[n]$ and W_N^{-k} complexes it is necessary to make 4 real multiplications and 4 real additions for the evaluation of each new value of $y_k[n]$ according to the system presented in Figure 2.

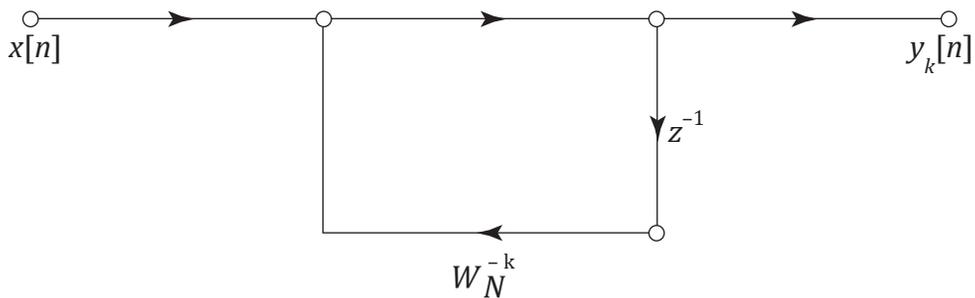


Figure 2. Flowchart of the complex recursive computation for the first-order filter of $X[k]$ [30].

As this process requires $4N$ real multiplications and $4N$ real additions to compute $X[k]$ it is less efficient, however, it avoids the storage and accumulation of the coefficients W_N^{kn} since they are calculated through the recursive form with

Equation (19) and the system of Figure 2. Additionally, it is possible to keep this reduction in a simplified way while reducing the number of multiplications by a factor of 2 with Equation (18) by multiplying the numerator and denominator by the factor $(1 - W_N^k z^{-1})$ resulting in Equation (20), as represented in Figure 3 with the flowchart that shows the direct development of the system function of the equation for a second order filter [23], [30].

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{(1 - W_N^{-k} z^{-1})(1 - W_N^{-k} z^{-1})}$$

$$H_k(z) = \frac{1 - W_N^k z^{-1}}{1 - 2\cos\left(\frac{2\pi k}{N}\right)z^{-1} + z^{-2}} \tag{20}$$

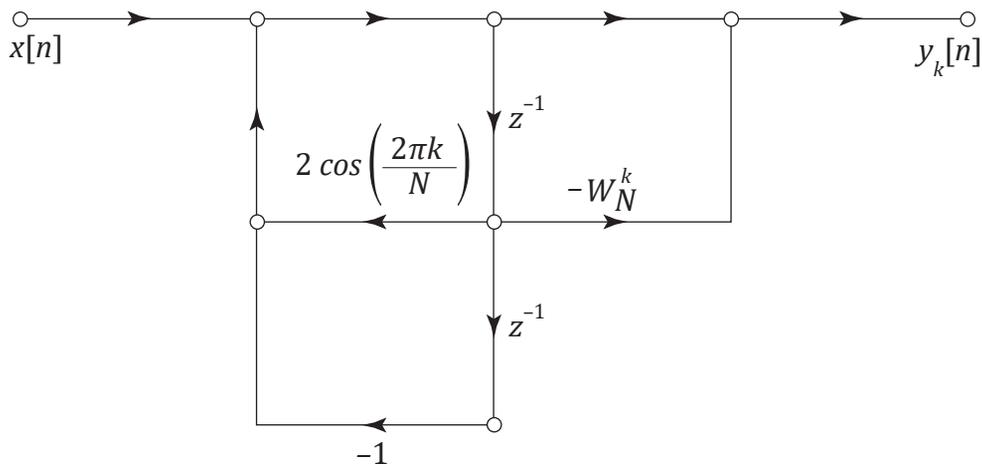


Figure 3. Flowchart of the complex recursive computation for the second order filter of $X[k]$ [30].

For a complex input, $2N$ real multiplications and $4N$ real additions are required for the poles of the system; while for the zero, 4 real multiplications and 4 other real additions are needed, resulting in a total of $2(N + 2)$ real multiplications and $4(N + 1)$ real additions [31], significantly reducing the number of multiplications required in the direct method. Therefore, the use of Goertzel's Algorithm ends up being an efficient method since it only requires the calculation and storage of the coefficients $\cos\left(\frac{2\pi k}{N}\right)$ and W_N^k [23], [32].

Finally, Equation (21) indicates the final value of the filter variable $y_k[n]$ which represents the DFT coefficient to be searched for $X[k]$ for the frequency k [26].

$$X[k] = y_k[n]|_{n=N}, \quad k = 0, 1, \dots, N \quad (21)$$

3. Methodology

To conduct this research, a literature search was performed using databases such as ScienceDirect, Google Scholar, IEEE, SpringerLink, and Scopus. The search focused on topics related to the Z-Transform, the Goertzel Algorithm, and their applications in engineering, as illustrated in Figure 4. The results were classified based on criteria such as conceptualization, properties, analysis, equations, and distinctive features of each study [33]. The review covered a time span from 2000 to 2024.

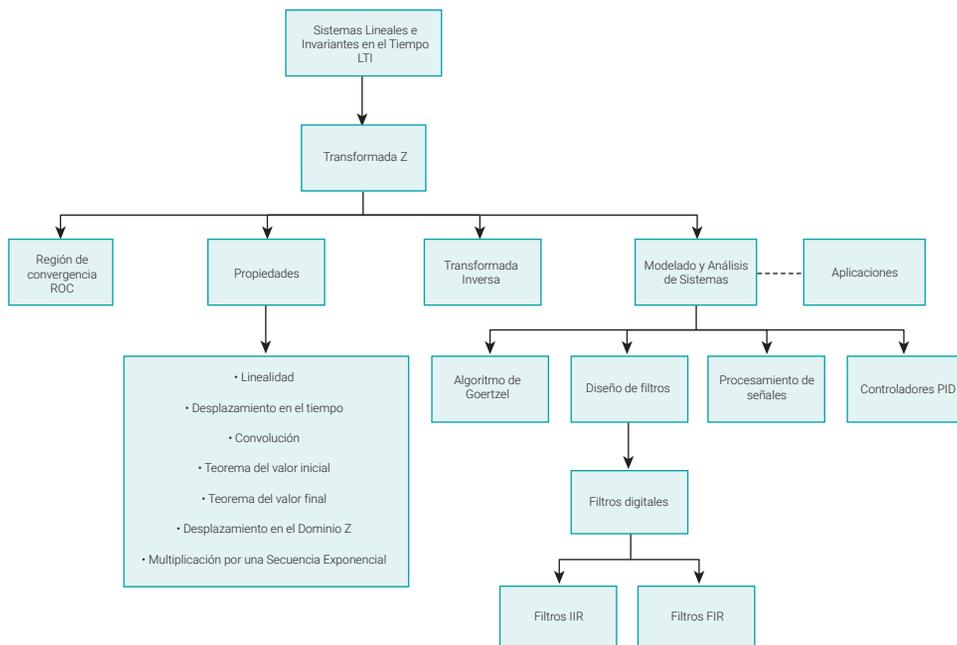


Figure 4. Categorization of the Z-Transform.

Source: own work.

The search results were filtered to identify the most relevant research, particularly in the field of telecommunications engineering. A taxonomic categorization and subcategorization were then performed, linking key engineering topics, including signal processing, biomedical processing, and control and automation systems.

Figure 4 presents topics related to the Z-Transform, including its properties, the Inverse Z-Transform, and its most significant theorems, as well as filters that utilize the Z-Transform and the Goertzel Algorithm.

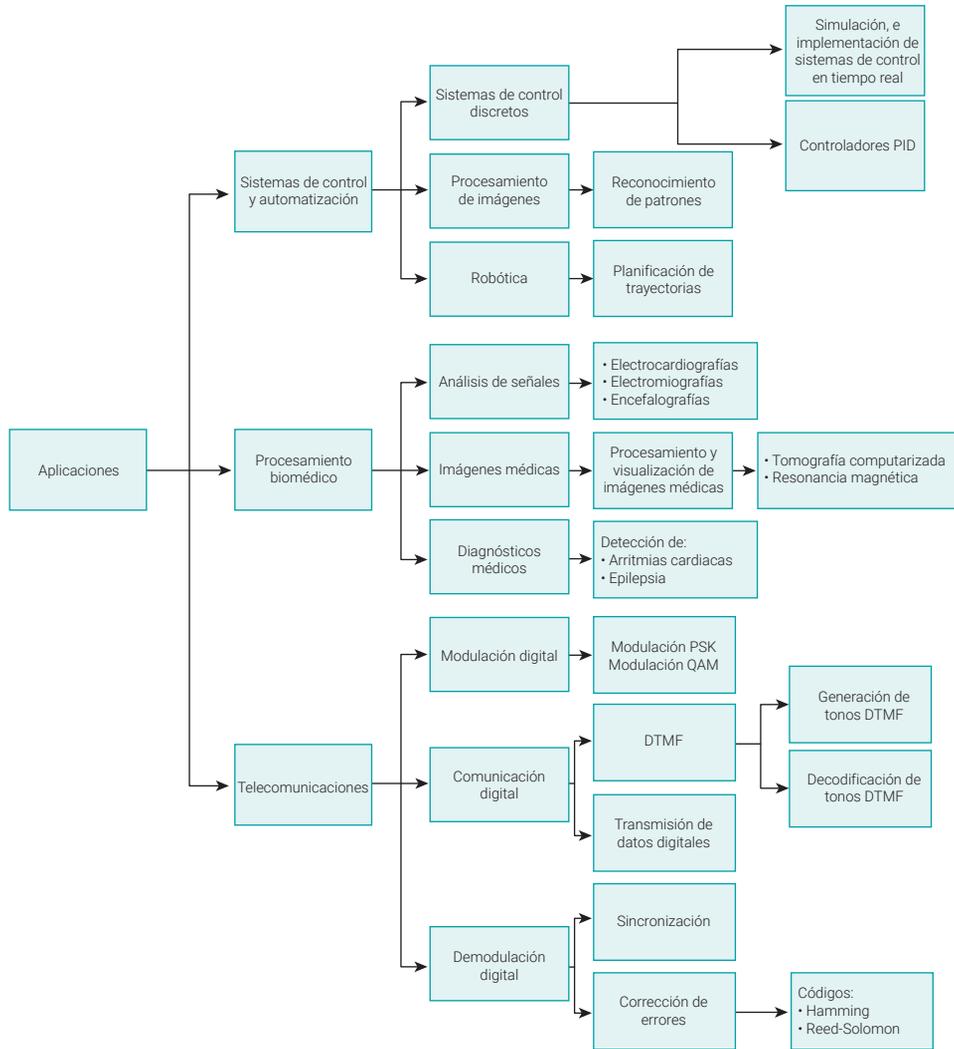


Figure 5. Categorization and subcategorization of Z-Transform applications in engineering.

Source: own work.

Figure 5 presents the classification of findings from the information search regarding the applications of the Z-Transform in various engineering fields, with a particular focus on telecommunications.

Subsequently, the study explores the applications of the Z-Transform in telecommunications through a case study on the detection of Dual-Tone Multi-Frequency (DTMF) tones in telephone network communications. This process involves transmitting dialed numerical digits by combining specific frequencies [34].

To illustrate this application, a MATLAB simulation is conducted, demonstrating the use of the Z-Transform alongside the Goertzel Algorithm for detecting all DTMF tones. The performance of the Goertzel Algorithm is compared with that of the Fast Fourier Transform (FFT) by measuring the simulation times in each case to determine which method is more efficient. The Goertzel Algorithm proves highly efficient for this application, as it enables precise analysis of discrete frequencies in audio signals and is widely used for decoding DTMF tones in telephone systems [35].

4. Z-transform in engineering

The Z-Transform has many fundamental uses in engineering, especially in the analysis and design of discrete systems. Some of the most relevant applications in which the Z-Transform is used are described below.

4.1. Control and automation systems

In control systems and signal processing, the Z-Transform is used in various applications that are implemented in discrete control systems highlighting the following, some are important in the design of PID controllers:

- Design and implementation of a discrete PID controller in programmable logic controllers [36],
- Gain tuning of a controller and analysis of its stability [37].
- Kalman filter implementation for navigation and object tracking systems [38].
- Digital audio processing of temporal signals with noise [39].
- Design of a tuner for PID controllers applied to first order electrical systems [40].
- Automation of a control system for the thermodynamic flow circuit of a parabolic trough solar collector [41].
- Robust, high-capacity audio watermarking based on the Z-Chirp Transform [42].

Likewise, there are multiple applications with filter design, among which the following stand out:

- Implementation of FIR-type digital filters on FPGA with ON-LINE reconfigurable coefficients [43].
- Mathematical modeling and simulation of an optimal linear adaptive FIR hybrid adaptive digital hybrid filter [5].
- Analysis of digital filters, audio or video compressors, and music synthesis [4].
- Use of the Discovery board for the calculation and implementation of FIR and IIR filters [6].

Also, other applications for the performance of real-time systems for motor control or vehicle control include:

- Induction Motor Speed Estimation by Rotor Slot Harmonic Frequency Using Zoom Enhanced Z-Chirp Transform Algorithm [44].
- Processing for control of autonomous vehicles [45].
- Linear optimal control for cooling of an internal combustion engine [46].
- Didactic platform for speed and position control of DC motors in Z-plane [47].

In robotics, the Z-Transform is applied in various areas, including the development of predictive control strategies for an inverted pendulum, which enables image stabilization in robots used for agricultural exploration [48]. This application, along with other agricultural-related topics, highlights the diverse uses of the Z-Transform.

4.2. Biomedical processing

In the field of medicine, the Z-Transform is used mainly in the analysis of signals for clinical examinations, among these applications the following were found:

- Accurate measurement of human heart rate based on multichannel radar data fusion [49].
- Stability of quantitative EEG parameters in boxing athletes [50].
- Comparison of zero-crossing detection techniques and the Z-Chirp Transform for measuring frequencies in the ultrasonic range. [51]

- Determination of hypertensive disease by Z-Chirp Transform and statistical characteristics of optimally bandpass-filtered short-duration photoplethysmography signals [52].

These applications also play a crucial role in medical diagnostics, as they enable a reliable reading of signals received by biomedical equipment used for patient monitoring and assessment.

Likewise, for the analysis of medical images, several investigations use the Z-Transform, among which some are indicated below:

- Nonlinear processing with Wavelet for noise removal in planar nuclear medicine images [53].
- Impulse response function of the intensity-curvature functional: applications in magnetic resonance angiography of the human brain [54].
- NIfTI neuroimage authentication using the elevation Wavelet Transform, Arnold Cat map, Z-Transform and Hessenberg decomposition [55].
- Hybrid compression of grayscale and color medical images using Quincunx wavelets and Walsh Hadamard Transform [56].

4.3. Telecommunications

In the field of telecommunications, there is a wide variety of applications of the Z-transform. It is mainly used to analyze and process discrete signals in the frequency domain for modulation, with implementations such as:

- Research on the identification of the modulation mode of PSK signal based on M-th power spectrum characteristics [57].
- Modulation of the software system of a radio station [58].
- Low level of complexity in two-stage FOE using modified Zoom-FFT for coherent optical M-QAM systems [59].
- Fourier and Z-Chirp transforms in the process of estimating horizontal and vertical synchronization frequency values of graphic displays [60].

In digital communication, the Z-Transform is also used to transmit digital data efficiently, sometimes using filters to optimize the transmission. This can be seen in the following applications:

- Coding of digital television audio and video signals [61].
- Signal processing and analysis of radar systems for [62].
- Design of a traffic flow controller in high-speed computer networks [63].
- Inverse filter design of an HF radio communication system [64].
- Contribution to near-field antenna measurement error cancellation by filtering techniques [65].
- A flow control algorithm for high-speed computer networks [66].

It is also used in the digital demodulation of signals to achieve data synchronization and error correction, among these applications are the following:

- Finite-length MMSE equalization complexity reduction using FFT [67].
- Efficient parallel architecture for a Reed-Solomon encoder with shared resources [68].
- Encrypted speech retrieval algorithm based on Z-Chirp Transform and feature extraction by perceptual hashing second [69].
- A Goertzel filter-based system for simultaneous and fast multifrequency EIS [70].

Another very important use of the Z-Transform in telecommunications is in telephony, specifically in tone dialing to separate the specific frequencies of the tones associated with each key.

4.3.1. DTMF

The Dual-Tone Multi-Frequency (DTMF) dialing system is a signaling method used in telecommunications to send information via audio signals on telephone lines. It is used in touch-tone phones to dial telephone numbers or perform various functions, such as navigating automated menu systems [71], [72].

DTMF was developed by Bell Labs in the 1960s, replacing rotary dials with pulse signaling, which was an error-prone and slow method [73]. Tone dialing became the standard for signaling on touch-tone telephones, as it was a more efficient and accurate method [74].

The system operates by generating audio tones, each composed of two specific frequencies. Each key on a keypad telephone is associated with a combination of two different frequencies: one frequency for each row and one frequency for each column, creating a unique combination for each dialed number [75], [76]. When a key

is pressed, tones of the two corresponding frequencies are generated simultaneously and transmitted over the telephone line.

The standard DTMF keypad has 16 keys, arranged in a 4x4 matrix as shown work in Table 1, where it is evident that the rows are assigned low frequencies (697, 770, 852, and 941 Hz), while the columns have high frequencies (1209, 1336, 1477, and 1633 Hz) [74].

Table 1. Keyboard with the combination of frequencies for DTMF tone generation.

	1209 Hz	1336 Hz	1477 Hz	1633 Hz
697 Hz	1	2	3	A
770 Hz	4	5	6	B
852 Hz	7	8	9	C
941 Hz	*	0	#	D

Source: own work.

Some advantages of DTMF are that it allows for faster dialing and has fewer errors. It is mainly used in telephone dialing, although it is also used for alarm controls, data phones, or in Interactive Voice Response (IVR) systems to navigate through automatic menus, such as customer service [31].

DTMF is currently being replaced in some of its applications by digital technologies, such as packet-based signaling in IP networks [77], [78]. However, DTMF remains a relevant technology and is used in many systems around the world [79], [80].

5. Case Study: Application of Goertzel's Algorithm for DTMF Tone Determination

We use the Goertzel Algorithm as a filter, which is defined in the transfer function presented above in equation (20), with the flow diagram shown work in Figure 3.

For this case study, we will compare the use of the Goertzel Algorithm and the FFT to detect DTMF tones according to the frequencies listed in Table 1. This will be done using a MATLAB simulation that illustrates and compares the efficiency of both methods.

First, we generate a signal with the DTMF frequencies of the desired key. In this case, it is for 'Key 1' (Figure 6), so we assign the frequency of the row and column corresponding to Key 1 (Table 1) to the variable "signal," with a sampling frequency of 8 kHz [81].

```
fm = 8000;
t = 0:1/fm:0.5;
fila = [697 770 852 941];
columna = [1209 1336 1477 1633];
senal = sin(2*pi*fila(1)*t)+sin(2*pi*columna(1)*t);
```

Figure 6. Generation of the DTMF Signal in MATLAB.

Source: own work.

Then the algorithm function is applied to the code (Figure 7); this function is already predefined in MATLAB with the implementation shown work in Figure 8 and is related to Equation (20).

```
N = length(senal);
k_row = round(fila * N / fm);
k_col = round(columna * N / fm);
goertzel_fila = zeros(1, length(fila));
goertzel_col = zeros(1, length(columna));
for i = 1:length(fila)
    goertzel_fila(i) = abs(goertzel(senal, k_row(i)));
end
for i = 1:length(columna)
    goertzel_col(i) = abs(goertzel(senal, k_col(i)));
end
goertzel_time = toc;
```

Figure 7. Calculation of Goertzel's Algorithm in MATLAB.

Source: own work.

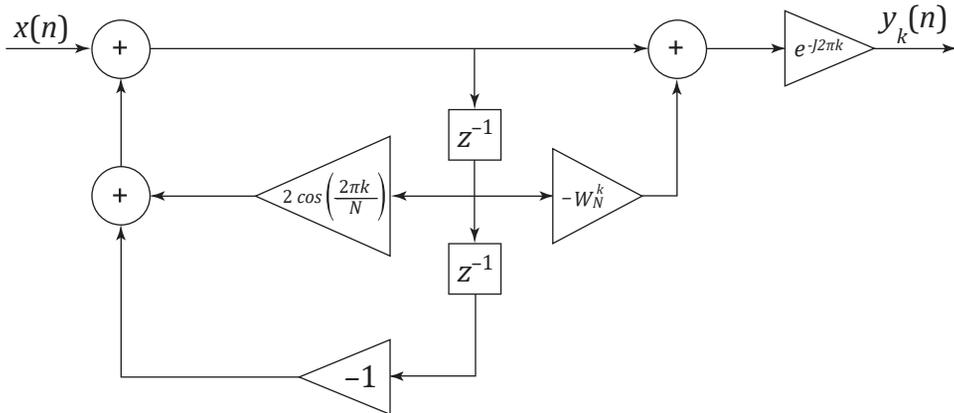


Figure 8. Diagram of the Goertzel Algorithm implemented in MATLAB [82].

Finally, we perform the FFT calculation according to Figure 9.

```
tic;
fft_result = abs(fft(senal));
fft_time = toc;
```

Figure 9. Calculation of the FFT in MATLAB.

Source: own work.

A measurement of the simulation time for the Goertzel Algorithm and the FFT was also performed. These values can be seen in the simulation in Figure 10, where the magnitudes of the row and column frequencies are observed, and it is verified that the implementation with the Goertzel Algorithm is more efficient because the time is shorter. For the case of Key 1, the time for Goertzel was 0.000406 seconds, while for the FFT it was 0.003117 seconds.

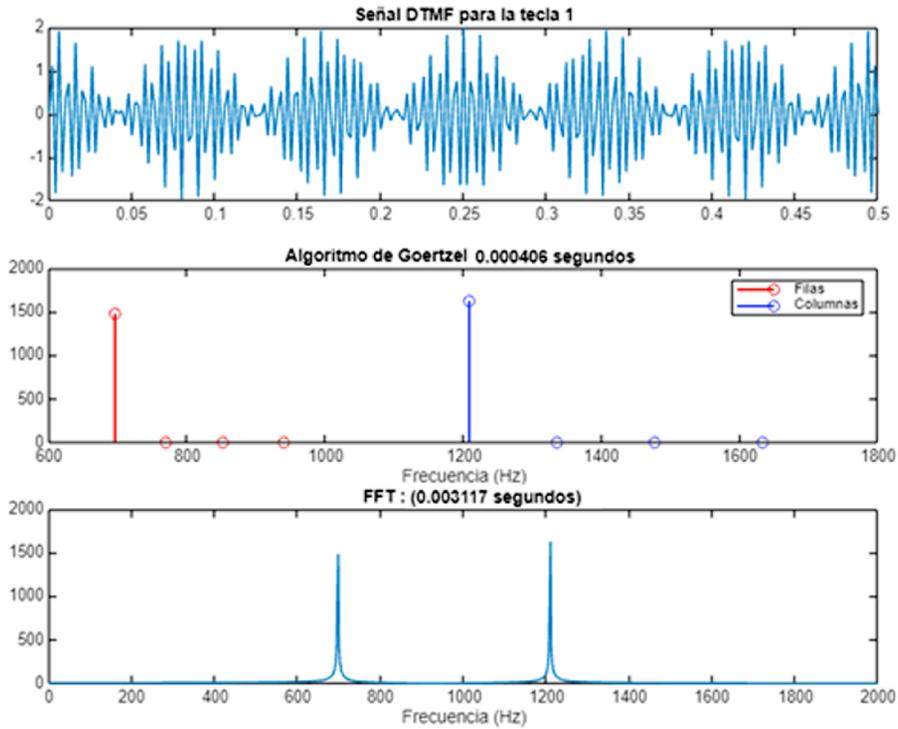


Figure 10. Simulation of the Goertzel Algorithm and FFT for DTMF for ‘Key 1’.
 Source: own work.

For the other DTMF keys, the simulation was also performed, and the shorter execution time in the simulation of the Goertzel Algorithm is noticeable, as shown work in Table 2, where the time values in milliseconds for the simulation with each key are presented.

Table 2. Results of the Goertzel Algorithm and FFT simulations for DTMF.

Key	Goertzel Algorithm Time (ms)	Time with FFT
1	0.406	3.117
2	0.409	2.771
3	0.494	2.589
4	0.412	2.728
5	0.583	2.93
6	0.407	2.77
7	0.38	2.562
8	0.394	2.676
9	0.386	2.673

(continúa)

(viene)

Key	Goertzel Algorithm Time (ms)	Time with FFT
*	0.377	2.872
0	0.436	2.824
#	0.382	2.764
A	0.399	2.734
B	0.379	2.718
C	0.397	2.706
D	0.381	2.744

Source: own work.

6. Conclusions

The Goertzel Algorithm is an efficient method that reduces the number of complex multiplications required to obtain each of the DFT coefficients compared to the direct DFT calculation. For complex inputs, it requires $2(N+2)$ multiplications and $4(N+1)$ real additions. It is ideal for applications where single-frequency detection is needed.

Research shows that the Z-Transform remains a basic tool in engineering. Its ability to analyze and design discrete systems in the frequency domain makes it unique for signal processing and data transmission.

A result of the implementation of the Goertzel Algorithm is the improvement in efficiency compared to the FFT in the case study. Considering the runtime values of each simulation, the result shows that the FFT is 7.6 times slower than Goertzel for the detection of DTMF tones.

When there are few specific frequencies, the Goertzel Algorithm can be useful, as in the case of DTMF, which uses 2 frequencies per key. However, if a wide range of frequencies is needed, the FFT is more useful, despite increasing the execution time and computational complexity, as it is a more mathematically complex process.

One of the most important conclusions is the efficiency of the Z-Transform in the analysis of the stability of discrete systems. This is because in continuous system implementations, there is noise reduction and fewer errors.

The review of Z-Transform applications with categorization contributes to future research on these topics and can be used as educational material in LTI systems.

Among the studies found, we reviewed some that propose new methods to increase the efficiency of the Goertzel Algorithm for complex mathematical computations and improve its runtime performance. These applications can be used to further optimize discrete systems.

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